Variational Calculations for Resonance Oscillations of Inhomogeneous Plasmas

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ABSTRACT

In this paper, the electrostatic resonance properties of an inhomogeneous plasma column are treated by application of the Rayleigh-Ritz method. In contrast to Parker, Nickel, and Gould (1964), who carried out an exact computation, we have used a description of the rf equation of motion and pressure term that allows us to express the system of equations in Euler-Lagrange form. The Rayleigh-Ritz procedure is then applied to the corresponding Lagrangian to obtain approximate resonance frequencies and eigenfunctions. An appropriate set of trial coordinate functions is defined, which leads to frequency and eigenfunction estimates in excellent agreement with the work of Parker, et al. (1964).

1. INTRODUCTION

This paper is concerned with the use of the Rayleigh-Ritz procedure to estimate the electron resonance frequencies of a warm inhomogeneous plasma column. This procedure has been extensively applied to single self-adjoint equations with great success (Mikhlin, 1964). For a system of equations, however, theoretical extensions have been noted only for the case of elliptic equations (Mikhlin, 1965). For the equations to be used here, which are not elliptic, it will be shown that accurate resonance frequencies can be predicted, provided that a certain set of coordinate functions is defined.

Previous theoretical treatments of the electron resonance problem give predictions which agree well with experimental data. However, these approaches have encountered difficulties stemming from the inhomogeneous electron density profile. For example, Parker, Nickel, and Gould (1964) solved numerically an appropriate fourth-order differential equation for the rf potential in a cylindrical positive column. Because of the exponential nature of the solutions in the cutoff region, their calculations were limited by the condition, $a^2/\overline{\lambda_D^2} \leq 4500$, where a is the column radius, $\overline{\lambda_D^2}$ $(= \epsilon_0 \kappa T/\overline{n_0} e^2)$ is the mean-squared Debye length, and $\overline{n_0}$ is the mean electron density in the column. Baldwin (1969) used a kinetic model in the low-temperature approximation to obtain the external admittance of a cylindrical plasma capacitor. The appropriate differential equation was solved after using inner-outer expansions connected through the resonance region. A WKB description was used for the inner region, where resonant waves are essentially evanescent, while a travelling wave description was used for the outer region, where Landau damping is important. Because the wave nature of the solutions was assumed in the outer region,

this theory is appropriate only for higher order resonances. Similar difficulties have been shown to occur in the simpler one-dimensional model (Harker, Kino, and Eitelbach, 1968; Miura and Barston, 1971; Peratt and Kuehl, 1972).

Variational methods offer an attractive alternative to these treatments. They have been used previously to estimate plasma resonance frequencies with simplified trial functions. Resonances of a cold inhomogeneous plasma were treated by Crawford and Kino (1963). Using the variational principle established by Sturrock (1958), Barston (1963) approximated the dispersion relations for wave propagation along an infinite cold plasma slab, and along the interface between two semi-infinite, counter-streaming cold plasmas. Some general features of the guided waves on a cold, transversely inhomogeneous plasma column in an axial magnetic field were studied by Briggs and Paik (1968). These papers (Crawford and Kino, 1963; Barston, 1963; Briggs and Paik, 1968) show that, with appropriate variational principles and judicious choices of trial functions, useful results can be obtained with relative ease by the variational approach.

A theoretical variational formulation for the electrostatic resonance oscillations of a warm, inhomogeneous plasma column in a dc electric or magnetic field of arbitrary direction was presented by Barston (1965) with the adiabatic index, γ, taken as unity. The variational principle to be presented here, however, is not restricted in the values of γ. One important feature in Barston's (1965) analysis is that the rf electric potential was treated as the solution of the rf Poisson equation, with the rf electron density considered given. It will be seen that the coordinate functions to be used here are defined in a similar fashion. A variational method of the Rayleigh-Ritz type has

been applied successfully by Dorman (1969) to a one-dimensional, warm, and field-free plasma with arbitrary dc density profile. A single second order differential equation for the electric field was obtained, and shown to have hermitian operators. The variational principle to be used here differs from Dorman's (1969) in that we are dealing directly with a system of Euler-Lagrange equations. In so doing, we can keep down the order of the equations, and are able to consider warm inhomogeneous plasmas in more than one dimension.

In this paper, we shall show that by appropriate definitions of the rf equation of motion and pressure term, the equations of the hydrodynamic model used by Parker, et al. (1964) become Euler-Lagrange equations. This will enable us to demonstrate the effectiveness of the Rayleigh-Ritz procedure in estimating the resonance frequencies of an inhomogeneous plasma. The associated numerical method mainly involves evaluations of definite integrals and solutions of finite algebraic eigenvalue equations, and is applicable over the entire range of $a^2/\overline{\lambda_D^2} \gg 1$ for estimating the first few resonance frequencies.

In most of the papers that deal with the electrostatic resonance problem (Crawford and Kino, 1963; Parker et al., 1964; Harker et al., 1968; Baldwin, 1969; Dorman, 1969; Miura and Barston, 1971; Peratt and Kuehl, 1972), it is assumed that the rf plasma current normal to the glass wall is zero. However, in the low temperature limit, $a^2/\overline{\lambda_D^2} \to \infty$, the main resonance frequency seems to agree with that of cold plasma theory, in which the normal rf plasma current is retained. We consider this problem and show that the resulting difference in predicted resonance frequencies is negligibly small for low pressure positive columns.

In §2, we present the basic equations, the corresponding Lagrangian, and the procedure to be applied in the variational approach. In §3, the numerical methods are explained before comparing computations with those of Parker et al. (1964). The paper concludes with a brief discussion in §4.

2. THEORY

For a low pressure positive column, moment equations with scalar pressure and negligible heat conduction are appropriate when the wave phase velocity is much larger than the thermal speed. For the first few electrostatic resonances, the wave phase velocity may be scaled to $\omega_p a$, where $\omega_p [=(e^2 n_0(0)/m\epsilon_0)^{1/2}]$ is the axial plasma frequency. Thus we require $a^2/\overline{\lambda_D^2} \gg 1$. A stationary ion background will be assumed, since we are interested only in electron resonances. Dissipation due to collisions, and Landau damping, will be neglected. Our analysis will consequently be valid only for the first few resonances. Also, the analysis will be quasi-static. Apart from some differences in definition of the rf equation of motion and pressure term, the equations we shall use are essentially those used by Parker, et al. (1964). The equations are generalized here to include dc magnetic field, $\underset{\sim}{B}_0$, and electron drift velocity, $\underset{\sim}{v}_0$. We have,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underset{\sim}{v}) = 0 , \qquad \epsilon_0 \nabla\cdot\underset{\sim}{E} + e(n-n_I) = 0 ,$$

$$mn\left(\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v}\right) + \nabla P + en\,(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) = 0 \qquad \left(\text{at } \underset{\sim}{r}_0 + \underset{\sim}{\xi}\right) . \quad (1)$$

Specialized to small perturbations, these reduce to the dc equations,

$$\nabla\cdot(n_0\underset{\sim}{v}_0) = 0 , \qquad \epsilon_0\nabla\cdot\underset{\sim}{E}_0 + e\,(n_0-n_I) = 0 ,$$

$$mn_0\underset{\sim}{v}_0\cdot\nabla\underset{\sim}{v}_0 + \nabla P_0 + en_0(\underset{\sim}{E}_0 + \underset{\sim}{v}_0 \times \underset{\sim}{B}_0) = 0 \qquad (\text{at } r_0) , \quad (2)$$

and rf equations,

$$\frac{\partial n_1}{\partial t} + \nabla\cdot(n_0\underset{\sim}{v}_1 + n_1\underset{\sim}{v}_0) = 0 , \qquad \epsilon_0\nabla\cdot\underset{\sim}{E}_1 + en_1 = 0 ,$$

$$mn_0\ddot{\underset{\sim}{\xi}} + (\nabla\cdot\underset{\sim}{\xi})\nabla P_0 + \nabla P_1 + \underset{\sim}{\xi}\cdot\nabla(\nabla P_0)$$

$$+ en_0\,(\underset{\sim}{\xi}\cdot\nabla\underset{\sim}{E}_0 + \underset{\sim}{E}_1 + \dot{\underset{\sim}{\xi}} \times \underset{\sim}{B}_0 + \underset{\sim}{v}_0 \times \underset{\sim}{B}_1) = 0 \quad (3)$$

where $\dot{\underset{\sim}{\xi}} = d\underset{\sim}{\xi}/dt = \partial\underset{\sim}{\xi}/dt + \underset{\sim}{v}_0 \cdot \nabla\underset{\sim}{\xi}$. In these equations, m and -e are the electron mass and charge; n and P are the electron density and pressure; $\underset{\sim}{E}$ is the electric field; n_I is the ion density; ϵ_0 is the vacuum permittivity; and $\underset{\sim}{\xi}$ is the perturbation displacement for the electrons (Figure 1).

The magnitude of $\underset{\sim}{v}_0$ is relatively small in the plasma region, but increases in the sheath region from the ion-acoustic speed to roughly the electron thermal speed at the glass wall (Self, 1963; Parker, 1963). We shall consequently neglect it in our analysis. However, due to the presence of non-zero $\underset{\sim}{v}_0$, $\underset{\sim}{n}_0$, and an rf electric field at the wall, a non-zero rf normal current term arises, and hence an rf surface charge term. The inclusion of this surface charge term, in the cases where the electron rf excursion exceeds the Debye length, is equivalent to the use of the dielectric model for a cold plasma column (Crawford, 1965). Further discussion of this surface charge term will be given in §3.4. With $\underset{\sim}{v}_0$ neglected, the following relations become appropriate

$$\begin{aligned} n_0(\underset{\sim}{r}) &= n_0(0) f(\underset{\sim}{r}) , \qquad f(\underset{\sim}{r}) = \exp[-e\varphi_0(\underset{\sim}{r})/\kappa T_e] , \\ n_1 &= -\nabla\cdot(n_0\underset{\sim}{\xi}) , \qquad \underset{\sim}{v}_1 = \partial\underset{\sim}{\xi}/\partial t , \end{aligned} \tag{4}$$

where κ is the Boltzmann constant, $\varphi_0(\underset{\sim}{r})$ is the dc column potential,

$$\underset{\sim}{E}_0(\underset{\sim}{r}) = -\nabla\varphi_0(\underset{\sim}{r}) , \tag{5}$$

and the first equation of (3) has been used to obtain n_1. The rf electron pressure, P_1 , is determined by the adiabatic equation of state,

$$P(\underset{\sim}{r}_0+\underset{\sim}{\xi})/P_0(\underset{\sim}{r}_0) = n(\underset{\sim}{r}_0+\underset{\sim}{\xi})^\gamma/n_0(\underset{\sim}{r}_0)^\gamma , \quad P_0 = n_0\kappa T_e . \tag{6}$$

The form of (6) can be understood by reference to Figure 1, and follows from the fact that when a cell is displaced from $\underset{\sim}{r}_0$ to $\underset{\sim}{r}_0 + \underset{\sim}{\xi}$,

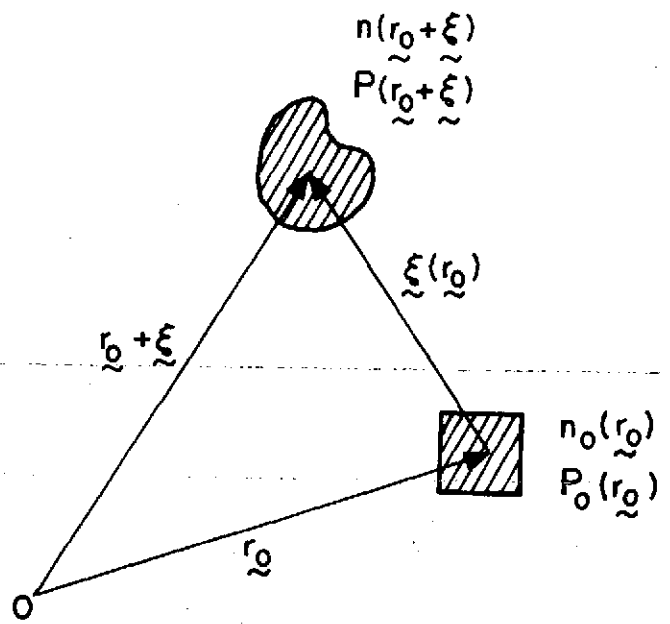

FIG. 1. Definition of plasma perturbation.

P is defined for the given cell, rather than for the given position, $\underset{\sim}{r}_0$. The adiabatic equation of state must consequently be applied to the same cell, before and after the displacement. Using the usual definition of perturbations,

$$P(\underset{\sim}{r}) = P_0(\underset{\sim}{r}) + P_1(\underset{\sim}{r}) , \qquad n(\underset{\sim}{r}) = n_0(\underset{\sim}{r}) + n_1(\underset{\sim}{r}) , \tag{7}$$

and (4), we obtain

$$P_1 = -\gamma P_0 \nabla\cdot\underset{\sim}{\xi} - \underset{\sim}{\xi}\cdot\nabla P_0 . \tag{8}$$

The rf force law in (3) is obtained by comparing the force laws in (1) and (2) in the same fashion (Newcomb, 1962).

It is now straightforward to use (4), (5), and (8) to rewrite (3) in terms of only $\underset{\sim}{\xi}$ and the rf potential, $\varphi_1 (-\nabla\varphi_1 = \underset{\sim}{E}_1)$,

$$\begin{aligned} & mn_0\ddot{\underset{\sim}{\xi}} - (\gamma-1)\nabla P_0\nabla\cdot\underset{\sim}{\xi} - \nabla\underset{\sim}{\xi}\cdot\nabla P_0 - \gamma P_0\nabla(\nabla\cdot\underset{\sim}{\xi}) \\ & \qquad - en_0(\nabla\varphi_1 + \underset{\sim}{\xi}\cdot\nabla\nabla\varphi_0 - \dot{\underset{\sim}{\xi}}\times\underset{\sim}{B}_0) = 0 , \\ & \epsilon_0\nabla^2\varphi_1 + e\nabla\cdot(n_0\underset{\sim}{\xi}) = 0 . \end{aligned} \tag{9}$$

Equations in (9) can be Fourier-transformed, normalized, and expressed in a cylindrical coordinate system, (r,θ), for a column of cylindrical symmetry,

$$\Omega^2 f\zeta_r - \Omega_c\Omega f\zeta_\theta + \Lambda_D^2\left[\gamma(\zeta_r' + \frac{1}{R}\zeta_r - \frac{\ell}{R}\zeta_\theta) + \frac{1}{R}(\ell\zeta_\theta - \zeta_r)\right] f'$$

$$+ \Lambda_D^2\,\gamma f(\zeta_r' + \frac{1}{R}\zeta_r - \frac{\ell}{R}\zeta)' + f(\Phi_0''\zeta_r + \Phi_1') = 0 , \tag{10}$$

$$\Omega^2 f\zeta_\theta - \Omega_c\Omega f\zeta_r + \Lambda_D^2 f'\frac{1}{R}(\ell\zeta_r - \zeta_\theta) + \Lambda_D^2\,\gamma f\frac{\ell}{R}\left(\zeta_r' + \frac{1}{R}\zeta_r - \frac{\ell}{R}\zeta_\theta\right)$$

$$+ f\left(\frac{1}{R}\Phi_0'\zeta_\theta + \frac{\ell}{R}\Phi_1\right) = 0 , \tag{11}$$

$$(R\Phi_1')' - \frac{\ell^2}{R}\,\Phi_1 + (Rf\zeta_r)' - \frac{\ell}{R}\,f\,\zeta_\theta = 0\ , \tag{12}$$

where the derivative with respect to R is denoted by $(')$ and

$$\Phi_0(0) = 0\ , \qquad \underset{\sim}{\zeta} = \hat{\underset{\sim}{i}}_r\,\zeta_r + \hat{\underset{\sim}{i}}_\theta(i\zeta_\theta)\ ,$$

$$\Phi_1(R,\theta,T) = \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} d\Omega\Phi_1(R)\exp\, i(\Omega T+\ell\theta)\ , \tag{13}$$

with $\hat{\underset{\sim}{i}}$ denoting unit vectors. The normalized quantities are defined as

$$R = r/a\ , \qquad T = \omega_p t\ , \qquad \Omega = \omega/\omega_p\ ,$$

$$\Omega_c = eB_0/m\omega_p\ , \qquad \underset{\sim}{\zeta} = \underset{\sim}{\xi}/a\ ,$$

$$\Lambda_D^2 = \lambda_D^2/a^2 = \epsilon_0\kappa T_e/n_0(0)e^2a^2\ , \qquad \Phi = \varphi\epsilon_0/n_0(0)ea^2\ , \tag{14}$$

and a static axial magnetic field, B_0, has been included.

2.1 Lagrangian Density

The forms of the force law in (3), and the rf pressure in (7), represent the important differences from the paper by Parker, et al. (1964) in that they make (9), as well as (10)-(12), systems of Euler-Lagrange equations without having to restrict the values of γ (Barston, 1965). In one of the models used by Dorman (1969), an appropriate pressure term similar to (7) was used without the benefit of the rf force law in (3). As a result, he was able to establish the variational principle only for the one-dimensional case.

The Lagrangian corresponding to (10)-(12) can be shown to be the following, by straightforward application of Hamilton's variational principle,

$$I_2 = \sum_{\ell=-\infty}^{\infty} \int_0^{\infty} d\Omega \mathfrak{L}_2(\Omega,\ell) \ , \qquad \mathfrak{L}_2(\Omega,\ell) = \Omega^2 A - \Omega_c \Omega B + H \ ,$$

$$A = \int_0^1 fRdR \left(|\zeta_r|^2 + |\zeta_\theta|^2 \right) \ , \qquad B = \int_0^1 fRdR \left(\zeta_r \zeta_\theta^* + \zeta_r^* \zeta_0 \right) \ ,$$

$$H = -\Lambda_D^2(\gamma T + T') - V_0 S - I + F + F' < 0 \ ,$$

$$T = \int_0^1 fRdR \left| \zeta_r' + \frac{1}{R} \zeta_r - \frac{\ell}{R} \zeta_\theta \right|^2 + \frac{1}{2} \left[f\zeta_r^* (\zeta_r' + \zeta_r - \ell\zeta_\theta) + c.c. \right]_{R=1} \ ,$$

$$T' = \int_0^1 fRdR \left[\frac{1}{R} \zeta_r' (\ell\zeta_\theta^* - \zeta_r^*) + \frac{1}{R} \zeta_\theta' (\ell\zeta_r^* - \zeta_\theta^*) + c.c. \right]$$

$$+ \frac{1}{2} \left[f\zeta_r^* (\ell\zeta_\theta - \zeta_r) + f\zeta_\theta^* (\ell\zeta_r - \zeta_\theta) + c.c. \right]_{R=1} \ ,$$

$$S = \int_0^1 fRdR \left(\phi_0'' |\zeta_r|^2 + \frac{1}{R} \phi_0' |\zeta_\theta|^2 \right) \ ,$$

$$I = -\int_0^1 fRdR \left(\zeta_r \Phi_1'^* + \frac{\ell}{R} \zeta_\theta \Phi_1^* + c.c. \right) \ , \qquad F = \int_0^1 RdR \left(|\Phi_1'|^2 + \frac{\ell^2}{R^2} |\Phi_1|^2 \right)$$

$$F' = \int_0^{R_b} \epsilon_g RdR \left(|\Phi_1'|^2 + \frac{\ell^2}{R^2} |\Phi_1|^2 \right) + \int_{R_b}^{R_c} RdR \left(|\Phi_1'|^2 + \frac{\ell^2}{R^2} |\Phi_1|^2 \right) \ ,$$

$$R_b = \frac{b}{a} \ , \qquad R_c = \frac{c}{a} \ , \qquad (15)$$

where c.c. denotes complex conjugate. Equation (15) is appropriate to the configuration shown in Figure 2 of a concentric metal cylinder surrounding a glass tube, of relative permittivity ϵ_g, that contains the plasma column. In the expressions for T and T′, the boundary terms are included to modify the natural boundary conditions on ζ_γ and ζ_θ (see for example, Courant and Hilbert, 1953) that would otherwith be unphysical. In the expressions for H and S, $\phi_0(r)$ and V_0 are defined as

$$\phi_0(r) = - \Phi_0(r)/V_0 , \quad V_0 = - \Phi_0(a) .$$

Substitution of the exact solutions of (10)-(12) would make $\mathfrak{L}_2(\Omega,\ell)$ zero. We see from (15) that, for negligible Ω_c, Λ_D, and V_0, the resonance frequencies are determined essentially by the values of I, F, and F′. Since V_0 is approximately proportional to Λ_D (Self, 1963; Parker, 1963), the effect of higher electron temperature is to raise each resonance frequency. When $\Omega_c \neq 0$, the roots of $\mathfrak{L}_2(\Omega,\ell) = 0$ are

$$\Omega_{1,2} = \Omega_c B/2A \pm \left[(\Omega_c B/2A)^2 - H/A\right]^{1/2} . \qquad (16)$$

Since Ω and $-\Omega$ are indistinguishable in experimental observations, we see that all the resonance frequencies are predicted by (16) to split in two, in agreement with the theoretical results of Barston (1965), and Vandenplas and Messiaen (1965). For sufficiently small Ω_c, when the values of B and A are not greatly affected by the presence of an axial static magnetic field, the amount of the split, $|\Omega_c B/A|$, will be proportional to Ω_c. This is in agreement with the observed splitting character of the main dipole resonance frequencies (Crawford, Kino, and Cannara, 1963; Messiaen and Vandenplas, 1962). Furthermore, since $A \geq |B|$, where the equal sign applies when $\zeta_\gamma \equiv \zeta_\theta$, this split is predicted to be always less than or equal to Ω_c.

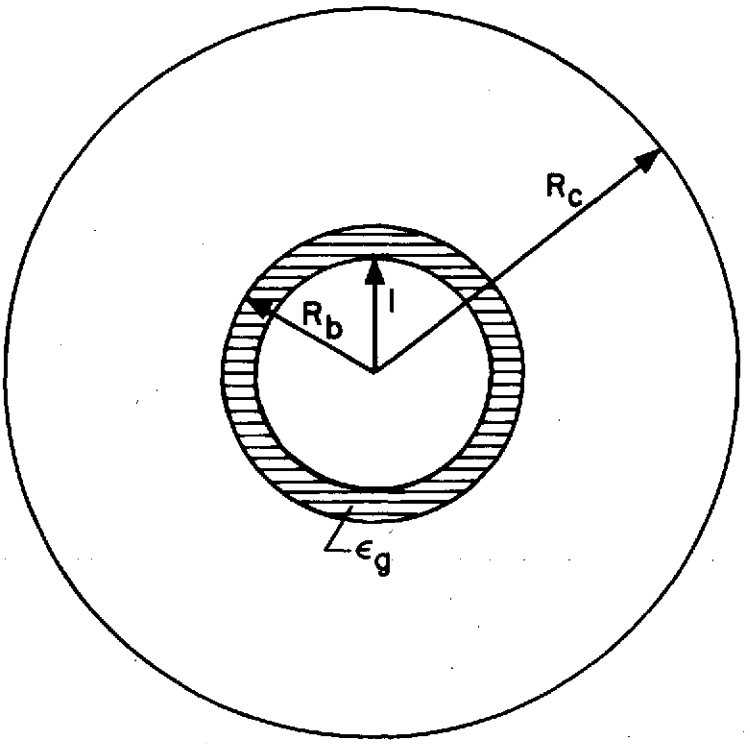

FIG. 2. Plasma column geometry.

2.2 Rayleigh-Ritz Procedure and Coordinate Functions

For a single linear Euler-Lagrangian equation, the Rayleigh-Ritz procedure is efficient in obtaining an approximate solution, by use of a weighted summation of a set of judiciously chosen coordinate functions. These coordinate functions must be linearly independent and complete, and satisfy the boundary conditions specified by the problem. The weighting coefficients that appear in the approximated Lagrangian are varied independently. This results in a system of algebraic equations that take the place of the original equation. Theoretically, better approximations can be obtained by using more coordinate functions. When eigenvalues are involved, the approximate eigenvalues always converge to the exact values from above (see for example, Mikhlin, 1964).

Much less attention has been given to the analogous problem for a system of linear Euler-Lagrangian equations, e.g. (10)-(12), though the theoretical extension of the variational method to a system of second order elliptic equations has been mentioned by Mikhlin (1965): the way to set up the corresponding coordinate functions is similar to that for the single equation case, i.e. the coefficients of each dependent variable are varied independently.

For our problem, in which (10)- (12) are not in elliptic form, the coordinate functions and the coefficients must be more restricted. In the Appendix to the paper it is shown that acceptable estimates of resonance frequencies and eigenfunctions can be obtained for our problem provided that the coordinate functions chosen for each dependent variable are related by (10)-(12).

By expanding ζ_r, ζ_θ, and Φ_1 in power series of R, and substituting in (10)-(12), we see that for small R, $\zeta_r \propto \zeta_\theta \propto R^{\ell-1}$, and $\Phi_1 \propto R^{\ell}$.

Thus, for $\ell \geq 1$, the solutions are well behaved at $R = 0$. If we also choose even functions for $f(R)$ and $\Phi_0(R)$, then ζ_r and ζ_θ are even in R, and Φ_1 is odd in R, when ℓ is odd, and vice versa. Since there are no other singularities in (10)-(12), polynomials in R constitute appropriate coordinate functions for our problem. For convenience, the coordinate functions chosen for ζ_r will be

$$\zeta_{rj} = R^{\ell-1} - R^{\ell+2j-1} \qquad (j = 1,2,\ldots) , \tag{17}$$

which conform to the usual assumption of zero normal rf current, since $\zeta_{rj}(1) = 0$.

Rather than choosing $\zeta_{\theta j}$ and $\Phi_{1,j}$ independently, we must determine them via the original differential equations and (17). After eliminating Φ_1 in (10) and (11), it follows by using the second equation of (4) that

$$R\Omega^2(R\zeta_\theta' + \zeta_\theta - \ell\zeta_r) = [R\,\Omega_c\Omega + \ell(\gamma-1)\Phi_0](R\zeta_r' + \zeta_r - \ell\zeta_\theta) . \tag{18}$$

With a given expression for $\Phi_0(r)$, and an assigned value of Ω, e.g. $\Omega = 1$, $\zeta_{\theta j}$ can then be easily determined for any given ζ_{rj}.

An immediate question arises concerning the dependence of the resulting variational estimate of resonance frequencies on the size of Ω chosen arbitrarily here. We have found that the first few resonance frequencies do not change by more than $10^{-4}\,\omega_p$ when Ω in (18) changes from 0.4 to 1 for all the values of $1/\Lambda_D^2$ used in this paper. If this were not the case, an iterative procedure would have to be used, i.e. the resulting variational estimate of Ω would have to be used in (18) to obtain a new set of $\zeta_{\theta j}$. These would be used in turn to obtain improved frequency estimates, and so on.

The corresponding coordinate function, $\Phi_{1,j}$, is obtained by solving (12) (Barston, 1965), with conditions of continuity of potential and normal displacement across the boundaries defined in Figure 2. According to the

discussion given in the Appendix, identical coefficients are assigned to each set of coordinate functions,

$$\zeta_r = \sum_{j=1}^{N} a_j \zeta_{rj} , \qquad \zeta_\theta = \sum_{j=1}^{N} a_j \zeta_{\theta j} , \qquad \Phi_l = \sum_{j=1}^{N} a_j \Phi_{l,j} , \tag{19}$$

before substitution in the Lagrangian, $\mathcal{L}_2(\Omega,\ell)$, of (15). The resulting Lagrangian then gives the algebraic Euler-Lagrange equation below,

$$\sum_{j=1}^{N} (\Omega^2 A_{ji} - \Omega\Omega_c B_{ji} + H_{ji}) a_j = 0 \qquad (i = 1,2,\ldots, N) ,$$

$$A_{ij} = \int_0^1 fRdR(\zeta_{ri}\zeta_{rj} + \zeta_{\theta i}\zeta_{\theta j}) , \tag{20}$$

where A_{ij}, B_{ij}, and H_{ij} are the matrix elements of the integrals, A, B, and H, given in (15), respectively, and are obtained by substituting the coordinate functions in a fashion given by the above A_{ij} expression. Equation (20) can be transformed into a generalized eigenvalue problem (Dorman, 1969)

$$\sum_{j=1}^{2N} (\Omega\bar{A}_{ij} - \bar{B}_{ij}) b_j = 0 \qquad (i = 1,2,\ldots, 2N) ,$$

$$b_j = (a_i, \Omega a_i) \qquad (j = 1,2,\ldots, 2N, \quad i = 1,2,\ldots, N) ,$$

$$\bar{A} = \begin{pmatrix} -H & 0 \\ 0 & A^+ \end{pmatrix} , \qquad \bar{B} = \begin{pmatrix} 0 & -H \\ -H^+ & \Omega_c B^+ \end{pmatrix} , \tag{21}$$

with superscript + signifying the transposition of a matrix. Equation (21) is now solvable by standard computer codes.

3. NUMERICAL METHODS AND RESULTS

The computing procedure is straightforward:

(i) read in physical and computational parameters,

(ii) compute coordinate functions $\zeta_{\theta j}$, $\Phi_{i,j}$, and all other functions appearing in the Lagrangian, $\mathfrak{L}_2(\Omega,\ell)$, at intervals Δr ,

(iii) compute $\mathfrak{L}_2(\Omega,\ell)$ by Simpson's rule to obtain A_{ij} , B_{ij} , and H_{ij} , and

(iv) solve (21) for Ω and a_j , and compute relevant eigenfunctions for N, N-1, and N-2 coordinate functions.

3.1. Approximate DC Density Profile

A density profile which approximates Parker's results (Parker, 1963), and is convenient for both analytical and numerical manipulation, is given by

$$f(R) = \exp[-\eta_w \phi_0(R)] , \qquad \phi_0(R) = \beta R^2 + (1-\beta)R^h , \tag{22}$$

where $h(>2)$ is an even integer, $\beta < 1$, and $\eta_w [= V_0/\Lambda_D^2]$ is the wall value of the potential function, $\eta(r)$, used by Self (1963) and Parker (1963). The particular form of (22) is used to justify the choice of coordinate functions of (17). Furthermore, with the use of (22), $\zeta_{\theta j}$ and $\Phi_{1,j}$ can now be solved analytically in terms of power series in R, in addition to the numerical solutions of (12) and (18). Comparison of the solutions by the two methods will provide an estimate of the degree of accuracy achieved in obtaining $\zeta_{\theta j}$ and $\Phi_{1,j}$.

The values of β and h are varied until f(r) best approximates, by least-square deviation, the profile given by Parker for specified values of Λ_D and η_w . The resulting profiles are shown in Figure 3, and the corresponding values of β and h are given in Table 1. It will be seen that as $1/\Lambda_D^2$ increases in value, (22) decreases in accuracy because

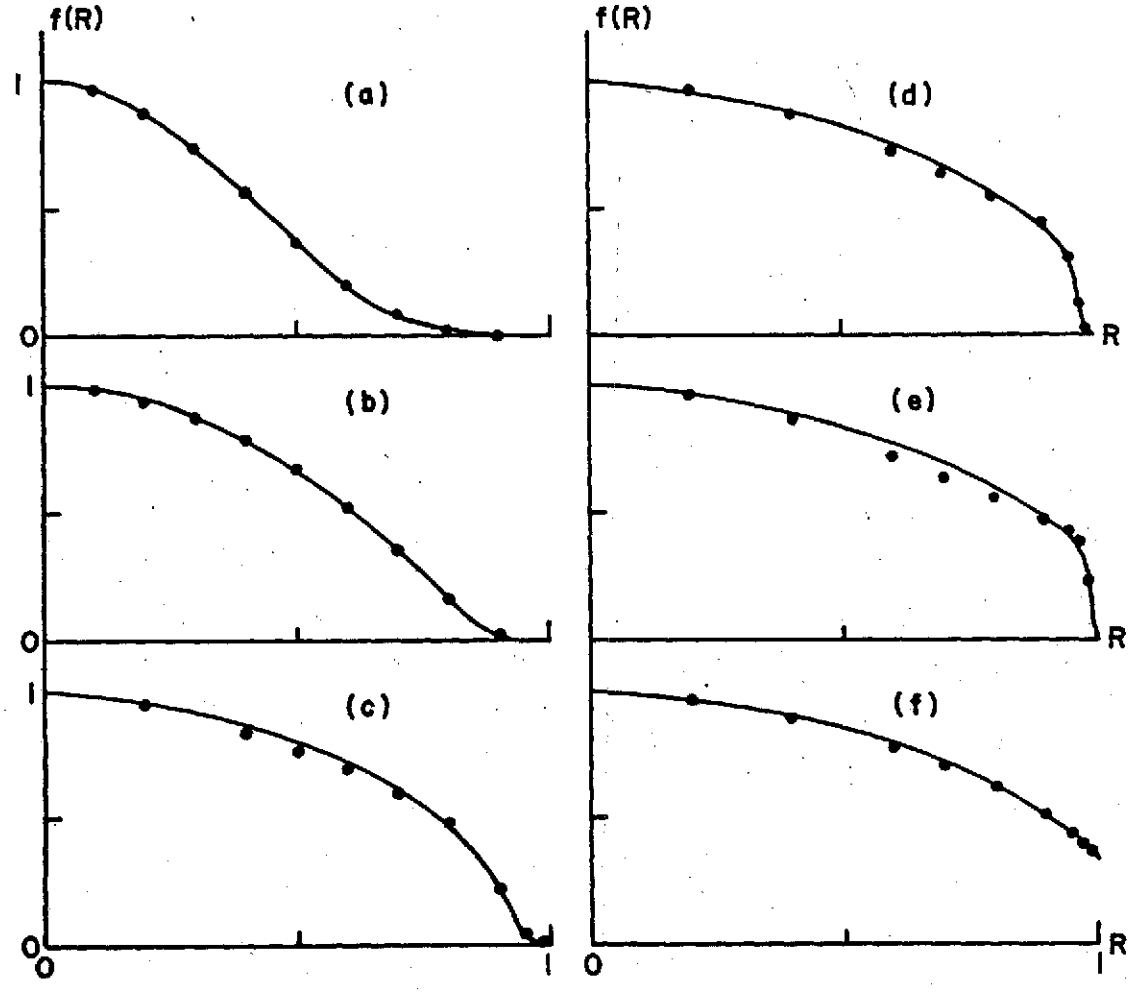

FIG. 3. Comparison between (22) (indicated by dots), and Parker's density profile, for the parameters given in Table 1.

Table 1

Parameters used in (22), and Figure 3, for a mercury-vapor positive column. The conversion between $1/\Lambda_D^2$ and $1\overline{\Lambda_D^2}$ is obtained by the calculations of Parker (1963).

	$1/\Lambda_D^2$	$1/\overline{\Lambda_D^2}$	η_w	β	h	$^*\Sigma\Delta^2$, † $\Sigma\Delta^2 R$
a	3.4×10^2	7.2×10^1	6.72	0.447	4	$^*6.2 \times 10^{-4}$
b	1.3×10^3	5.1×10^2	6.60	0.221	8	$^*1.3 \times 10^{-4}$
c	8.2×10^3	4.5×10^3	6.52	0.159	20	$\dagger 6.2 \times 10^{-4}$
d	6.7×10^4	4.3×10^4	6.44	0.142	54	$\dagger 2.9 \times 10^{-3}$
e	6.2×10^5	4.2×10^5	6.40	0.141	156	$\dagger 7.5 \times 10^{-3}$
f	∞	∞	1.08	0.644	12	$\dagger 1.3 \times 10^{-4}$

of the increasing degree of steepness displayed by the density profile in the sheath region. At zero temperature, the sheath is omitted and an accurate approximation can again be obtained. The approximation of (22), however, was found to be sufficient to give accurate frequency and eigenfunction estimates for all of the values of $1/\Lambda_D^2$ listed in Table 1.

3.2. Solutions for $\zeta_{\theta j}$ and $\Phi_{1,j}$

By the use of (14), (15), and (19), $\zeta_{\theta j}$ can be put in the form of a rapidly converging series,

$$\zeta_{\theta j} = R^{\ell-1} - \sum_{i=1}^{\infty} \eta_{j+1,i} R^{\ell+2j+(i-1)(h-2)-1} ,$$

$$\eta_{j,1} = \frac{\Omega^2+(\ell+2j-2)\left[\frac{\Omega_c \Omega}{\ell} - 2\beta(\gamma-1)V_0\right]}{\frac{\Omega^2}{\ell}(\ell+2j-2) + \Omega_c\Omega - 2\beta(\gamma-1)\ell V_0} ,$$

$$\eta_{j,2} = \frac{(\gamma-1)(1-\beta)hV_0[\ell a_{j,1} - (\ell+2j-2)]}{\frac{\Omega^2}{\ell^2}(\ell+2j+h-4)+\Omega_c\Omega-2\beta(\gamma-1)\ell V_0} ,$$

$$\eta_{j,i} = \frac{\ell^2(\gamma-1)(1-\beta)hV_0}{i\Omega^2(h-2)} \eta_{j,i-1} \qquad (i \geq 3). \tag{23}$$

Since the maximum value of V_0 of interest is roughly 0.02 and $\Omega \sim 1$, one needs at most five or six terms in (23) to attain a precision of 10^{-8} for $\zeta_{\theta j}$.

To solve for $\Phi_{1,j}$, the predictor-corrector method of Adams-Bashforth (Fox, 1962) has been used on (12), which can be reduced to the form,

$$y_1' = y_2/R , \qquad y_2' = g(R) + \ell^2 y_1/R ,$$

$$y_1 = \Phi_{1,j} , \qquad y_2 = R\Phi_{1,j}' ,$$

$$g(R) = f(-R\eta_w \Phi_0' \zeta_{rj} + R\zeta_{rj}' + \zeta_{rj} - \ell\zeta_{\theta j}) . \tag{24}$$

Since the complementary solution of (24) is $\Phi^N_{l,j} = y^N_l = R^{\ell}$, and its particular solution, y^P_l, is proportional to $R^{\ell+2j}$ near $r = 0$, $y^{P'}_l(0)$ and $y^{P'}_2(0)$ are both equal to zero. So the starting values of y^P_l and y^P_2 are well-behaved, and easily obtained by Taylor series expansion near $R = 0$. The total solution of $\Phi_{l,j}$ can then be written as

$$\Phi_{l,j} = c_j R^{\ell} + \Phi^P_{l,j} , \tag{25}$$

where c_j is determined by imposing the boundary conditions of Φ_l. By making the interval $\Delta r = 0.01$, and using double precision, $\Phi_{i,j}$ can be calculated to within 10^{-8}. This is arrived at, first, by comparing results that use different values of Δr, and secondly, checking against solutions of (12) obtained by power-series expansions in R.

3.3 Numerical Instability

In the process of solving the algebraic equation, (21), the size of N is limited by the inaccuracy involved in obtaining A_{ij}, etc. This inaccuracy introduces a numerical instability whenever the coordinate functions are not orthogonal functions with respect to the differential operators of (10)-(12) (Mikhlin, 1971, Chap. 2). The situation is best illustrated by an example in which the dipole resonance frequencies corresponding to Figure 3(b) are calculated for different values of N while holding the size of Δr constant at 0.05.

As shown in Table 2, as N is increased from 2, the first few resonance frequencies are approached from above with rapidly stabilized estimates. When N is increased beyond 8, undesirable fluctuations larger than 10^{-4}, and clearly erratic changes in the values of Ω, start to appear. In the case $N = 9$, for example, one would obtain an erroneous

Table 2

Dipole resonance frequency estimates obtained with $\Delta r = 0.05$ for the case of Figure 3(b). Significant numerical instability sets in when $N \geq 8$. The 'best estimates' are obtained with $\Delta r = 0.01$ and $N = 9$.

N	Main	First	Second	Third	Fourth
Best estimates:	0.4447	0.7099	0.8746	0.9924	1.080
2	0.4703	0.8813			
3	0.4486	0.7446	0.9250		
4	0.4450	0.7155	0.8818	1.022	
5	0.4448	0.7119	0.8757	0.9958	1.084
6	0.4448	0.7117	0.8753	0.9956	1.084
7	0.4448	0.7104	0.8751	0.9947	1.082
8	0.4448	0.7098	0.8748	0.9921	1.081

fundamental resonance frequency. Characteristic of the variational nature of the Lagrangian, $\mathcal{L}_2(\Omega,\ell)$, more serious errors are found in the approximate eigenfunctions, than in the resonance frequencies. The optimal combination of N and Δr, that produces acceptable results in the shortest computation time, can be obtained by trial and error. Repeated solution of (21) for a few adjacent values of N thus becomes an economical technique: this requires computation of the matrices A_{ij}, etc. only once, and offers safeguards against obtaining erroneous results due to numerical instabilities.

3.4 Computer Results

As a practical example, the approximate density profiles given in Table 1 have been used to predict dipole resonances for Tube No. 1 used by Parker, et al. (1964) ($\ell = 1$; $a = 0.5$ cm; effective relative permittivity at the surface of the column $K_{eff} = \Phi_1'/\ell\Phi_1 = 2.1$). The computation time varies roughly as N^2. With $N = 10$, a typical calculation takes about 40 seconds, and requires a core space of less than 100K bytes in an IBM 370/67 machine.

The resulting approximate solutions for $\xi_r, i\xi_\theta, n_1, \varphi_1$, and φ_1' are plotted in Figures 4-6. The density and the radial electric field solutions, $n_1(R)$ and φ_1' resemble very closely those given by Parker, et al. (1964), and Parbhakar and Gregory (1971), respectively. The relative amplitudes of ξ_r and ξ_θ shown in these figures are retained, revealing that as $1/\overline{\Lambda_D^2}$ increases, ξ_r progressively dominates over ξ_θ. For $1/\overline{\Lambda_D^2} > 4500$, it will be seen that the perturbations should be progressively compressed toward the sheath region as $1/\overline{\Lambda_D^2}$ increases in value. These

solutions are not reproduced here because they also exhibit undesirable oscillations with wavenumber equal to N, an expected characteristic when we try to approximate rapidly varying functions with truncated polynomials. Since only a moderate computer storage is used for $N \leq 10$, there is room to increase N, and decrease Δr, to obtain better approximate solutions. However, this is considered unimportant for our purpose, since we are able to obtain good frequency estimates for this region with $N \leq 10$, as Figure 7 reveals.

The corresponding estimates of resonance frequencies are shown in Figure 7. Since the electron temperature corresponding to the experimental resonance data of Parker et al. (1964) was adjusted to fit their theoretical spectrum, it would be reasonable for us to make a similar adjustment. As is evident from Figure 7, however, no such adjustment is necessary. Indeed, our result seems to be in slightly better agreement with the $T_e = 3$ eV data. The minor differences between the two theoretical results probably come from the differences in the rf equation of motion and pressure term used in the two treatments

Similar to other papers (Crawford, 1964; Parker et al. 1964; Harker et al. 1968; Baldwin, 1969; Dorman, 1969; Miura and Barston, 1971; Peratt and Kuehl, 1972), we have assumed zero normal rf plasma current density at the glass wall, through the form of ζ_{rj} in (17). This is appropriate when the plasma is sufficiently warm that the electron excursion velocity is much smaller than the thermal speed, and $f(1) \ll 1$. This assumption, however, is inconsistent with the dielectric model for a cold plasma column, where normal rf plasma current must be included. It is of interest to ask why the main resonance frequency of a warm plasma column,

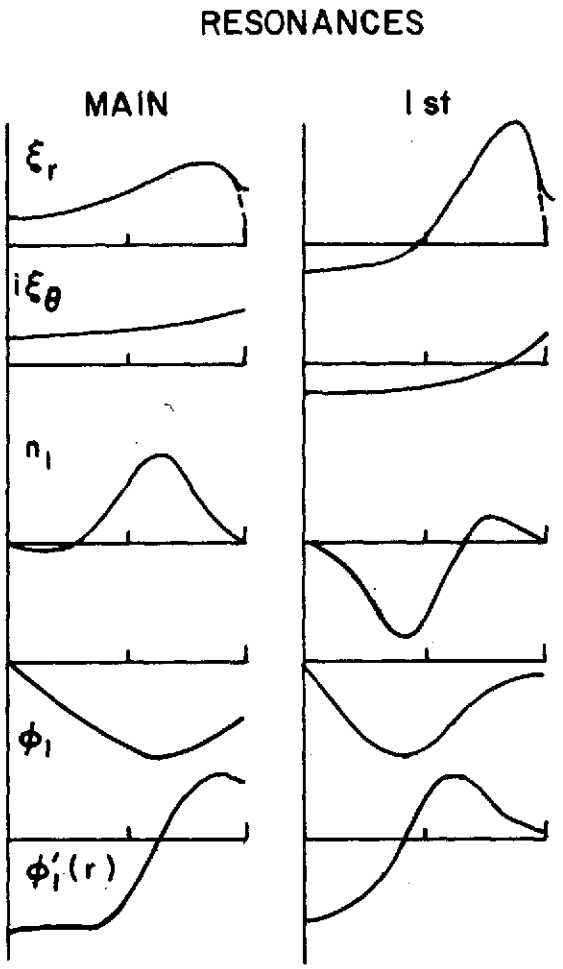

FIG. 4. Approximate solutions for $1/\overline{\lambda_D^2} = 72$, $\ell = 1$.

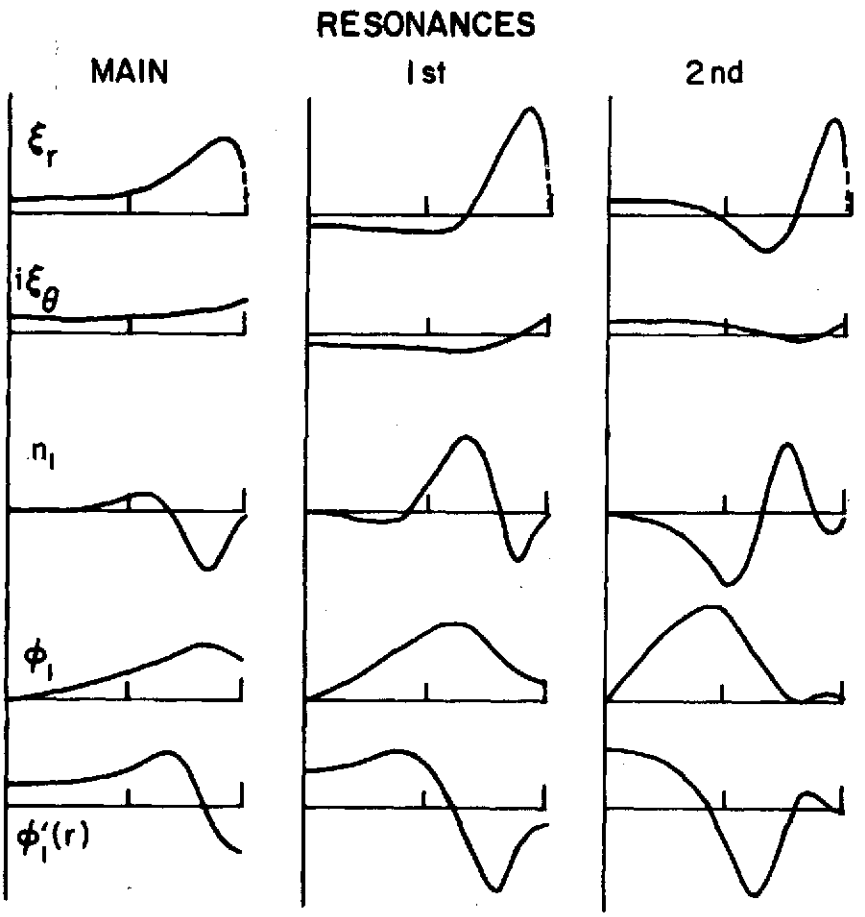

FIG. 5. Approximate solutions for $1/\lambda_D^2 = 510$, $\ell = 1$.

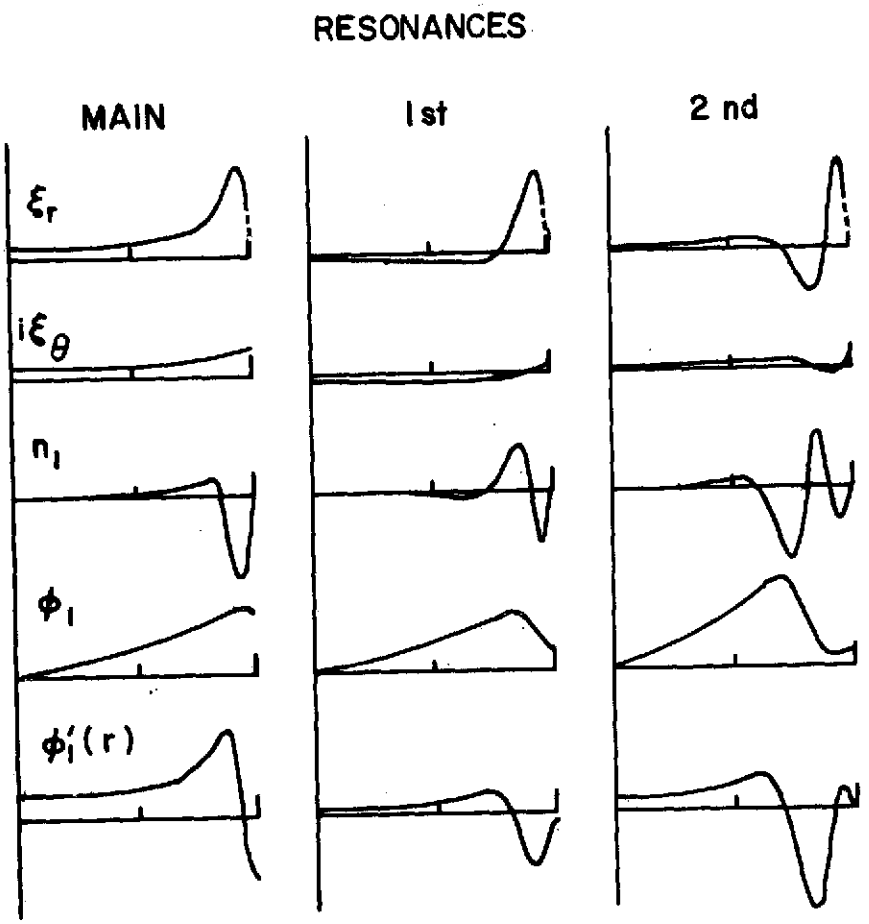

FIG. 6. Approximation solution for $1/\overline{\lambda_D^2} = 4500$, $\ell = 1$.

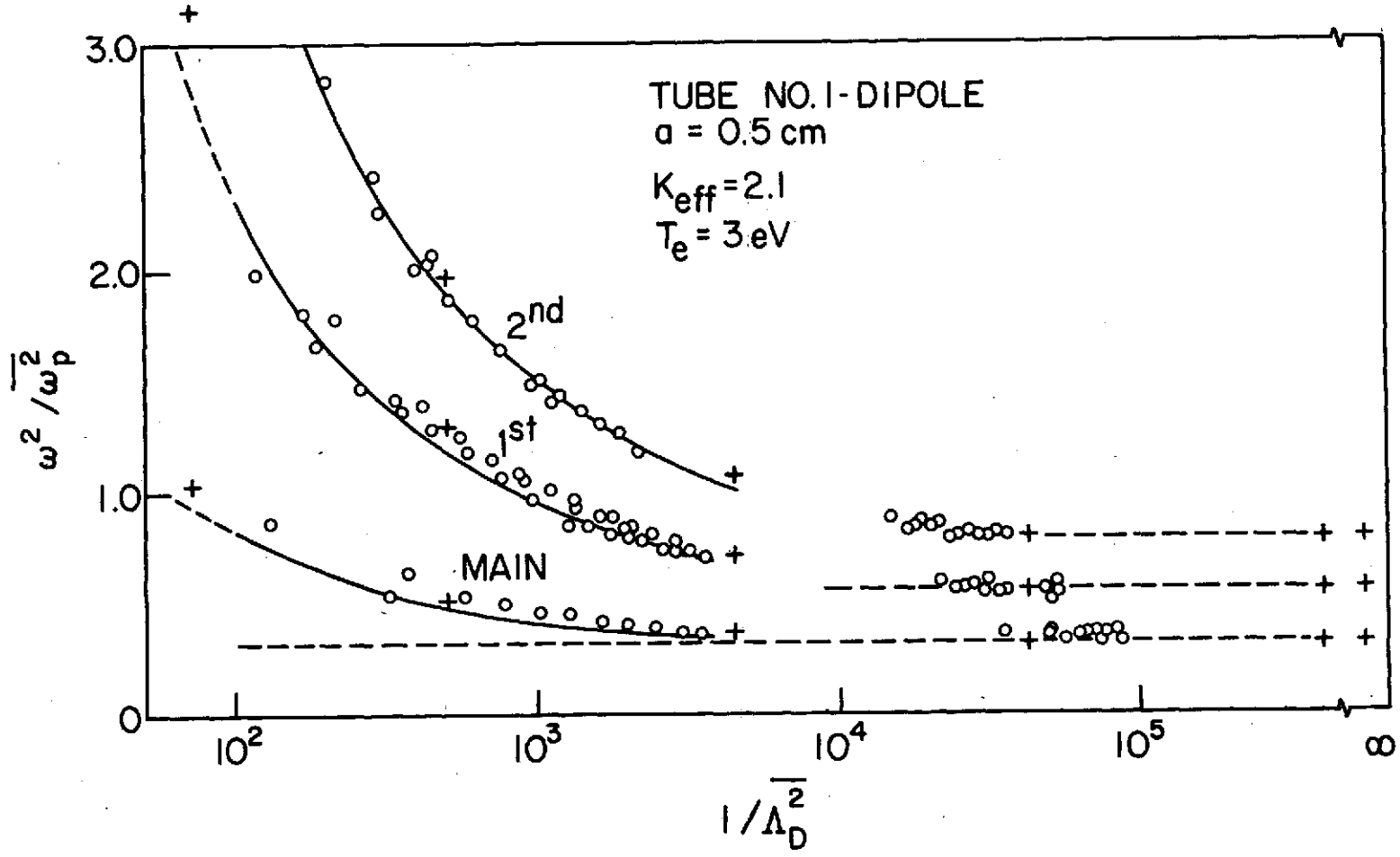

FIG. 7. Dipole resonance spectrum of Tube No. 1 (Parker et al.), compared with estimates by the variational method (indicated by crosses).

in the limit of low temperature, approaches the principal resonance of a cold plasma column. To answer this question, we need only change (17) to

$$\zeta_{rj} = R^{\ell-1} + R^{\ell+2j-1} \qquad (j = 1,2,\ldots) , \tag{26}$$

and impose the requirement of continuity of normal displacement in the form

$$\Phi_1'(1^-) + f(1)\zeta_r(1) = \epsilon_g \Phi_1'(1^+) . \tag{27}$$

The resulting solutions of ζ_r are found to be only slightly different from the previous case near $R = 1$ [Figures 4-6, where the dashed lines correspond to the use of (17)]. Furthermore, the main resonance is lowered by less than 1 per cent for all of the values of $1/\overline{\Lambda_D^2}$ used here, including the case $1/\overline{\Lambda_D^2} \to \infty$. This is well within the experimental errors.

4. DISCUSSION

In this paper, we have applied the Rayleigh-Ritz procedure to a system of three Euler-Lagrangian equations that describe the electron resonances of a nonuniform warm plasma column. It is shown that accurate frequencies for the first few resonances can be obtained for the entire range of $1/\overline{\Lambda_D^2} \gg 1$. Results which agree closely with those of Parker, et al. (1964) have been obtained

Contrary to the case of a system of elliptic equations, where the coefficients are assigned independently to each dependent variable (Mikhlin, 1971), we have found that for (10)-(12), the same coefficient must be assigned to each set of coordinate functions, e.g. (19). In addition to the usual requirements, that the coordinate functions must be linearly independent and complete, we have chosen that they be set up in accordance with (10)-(12).

The present method can be easily modified to include the effects of electron dc drift, dc magnetic field, and ion motion. With the axial dimensions and rf magnetic field included, this procedure would be efficient in solving travelling wave problems in a nonuniform plasma waveguide.

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REFERENCES

Baldwin, D. E. 1969, Phys. Fluids, 12, 279.

Barston, E. M. 1963, Phys. Fluids, 6, 828.

Barston, E. M. 1965, Phys. Rev., 139, A394.

Briggs, R. J. and Paik, S. F. 1968, Int. J. Electron., 23, 163.

Courant, R. and Hilbert, D. 1953, Methods of Mathematical Physics, Vol. 1, Interscience, p. 208.

Crawford, F. W., Kino, G. S. 1963, C. R. Hebd. Seanc. Acad. Sci., Paris, 256, 1939 and 2798.

Crawford, F. W., Kino, G. S. and Cannara, A. B. 1963, J. Appl. Phys., 34, 3168.

Crawford, F. W. 1964, J. Appl. Phys., 35, 1365.

Crawford, F. W. 1965, Int. J. Electron., 19, 217.

Dorman, G. 1969, J. Plasma Phys., 3, 387.

Fox, L. 1962, Numerical Solution of Ordinary and Partial Differential Equations, Addison-Wesley, p. 29.

Harker, K. J., Kino, G. S., and Eitelbach, D. L. 1968, Phys. Fluids, 11, 425.

Messiaen, A. M. and Vandenplas, P. E. 1962, Physica, 28, 537.

Mikhlin, S. G. 1964, Variational Methods in Mathematical Physics, Pergamon.

Mikhlin, S. G. 1965, The Problem of the Minimum of a Quadratic Functional, Holden-Day, p. 148.

Mikhlin, S. G. 1971, The Numerical Performance of the Variational Method, Wolters-Noordhoff.

Miura, R. M. and Barston, E. M. 1971, J. Plasma Phys., 6, 271.

Newcomb, W. A. 1962, Nucl. Fusion, Supplement, Part 2, 451.

Parbhakar, K. J. and Gregory, B. C. 1971, Can. J. Phys., 49, 2578.

Parker, J. V. 1963, Phys. Fluids, 6, 1657.

Peratt, A. L., Kuehl, H. H. 1972, Phys. Fluids, 15, 1117.

Parker, J. V., Nickel, J. C., and Gould, R. W. 1964, Phys. Fluids, 7, 1489.

Self, S. A. 1963, Phys. Fluids, 6, 1762.

Sturrock, P. A. 1958, Ann. Phys., 4, 306.

Vandenplas, P. E. and Messiaen, A. M. 1965, Nucl. Fusion, 5, 47.

APPENDIX

Here we shall show that the coordinate functions, ζ_{rj} , ζ_{0j} , and $\Phi_{1,j}$, must be assigned the same coefficients, a_j , for the Rayleigh-Ritz procedure applied to $\mathcal{L}_2(\Omega,\ell)$ of (15) to be successful: it will be shown that, by restricting these coordinate functions according to (10)-(12), the appropriate Rayleigh-Ritz procedure can be established.

A system of second order elliptic equations can be written as

$$(D_{jk}U_k')' + \lambda E_{jk}U = 0 , \quad D_{jk} = D_{kj} , \quad E_{jk} = E_{kj} , \tag{A.1}$$

where U_k is the k^{th} dependent variable, the matrices D_{jk} and E_{jk} are functions of the independent variable x , the eigenvalue is $\lambda(\geq 0)$, and the summation convention has been used. Ellipticity demands that

$$D_{jk}\alpha_j\alpha_k > 0 , \quad E_{jk}\alpha_j\alpha_k \geq 0 , \tag{A.2}$$

for any real non-zero vector α_j . The solutions of (A.1) then admit of variational estimates, as outlined by Mikhlin (1965).

When we apply the Rayleigh-Ritz procedure to the Lagrangian for (A.1),

$$L = \int_0^1 dx \, (D_{jk}U_j'U_k' - \lambda E_{jk}U_jU_k) , \tag{A.3}$$

the coefficients preceding the coordinate function for each U_k can be varied independently. Suppose a set of legitimate trial functions, $U_k = C_kV_k$ (summation convention not used here) with coefficient C_k , are used in (A.3). Because of (A.2), we can obtain crude estimates of λ larger than the lowest eigenvalue, even if only one of the C_k is non-zero.

If the same procedure is applied to the Lagrangian in (15), we will obtain erroneous estimates of Ω . Consider the case of a cold, uniform plasma column with $B_0 = 0$, so that $H = -I + F + F'$. Suppose C_r ,

C_θ , and C_ϕ are the coefficients for the coordinate functions ζ_{rj} , $\zeta_{r\theta}$, and $\Phi_{i,j}$, respectively. Then if $C_\phi = 0$, the value of the Ω estimate will be zero, and fail the requirement of the Rayleigh-Ritz sequence.

Consequently, to make $(-H/A)^{1/2}$ from (15) at least non-zero, we must use a single coefficient for each set of the coordinate functions ζ_{rj} , $\zeta_{\theta j}$, and $\Phi_{1,j}$. Merely making $C_r = C_\theta = C_\phi = a_j$ is insufficient to produce a legitimate Rayleigh-Ritz sequence, because the value of $\Phi_{1,j}$, for example, can be arbitrarily small in comparison with ζ_{rj} and $\zeta_{\theta j}$, making the Ω estimates also arbitrarily small. We see that restricting these coordinate functions according to (10)-(12), is sufficient to reduce the resulting approximated Lagrangian to a single Euler-Lagrange equation. The appropriateness of the resulting Rayleigh-Ritz procedure can then be guaranteed. It is conceivable, of course, that there may be less restrictive choices of appropriate coordinate functions corresponding to the Lagrangian in (15), but we have not chosen to pursue this point.